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Stability of Parisi's solution of a spin glass model

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Abstract. The condition of stability of Parisi's solution of the infinite-range spin glass model is obtained and examined near the transition temperature. The matrix of second derivatives $\partial^2 F/\partial Q_{\alpha\beta}\,\partial Q_{\gamma\nu}$ has non-negative eigenvalues only. There are two continuous branches of the eigenvalues and zero is an accumulation point of the eigenvalues.

1. Introduction

The problem of formulating a convincing mean-field theory of spin glasses has attracted much interest in recent years. Sherrington and Kirkpatrick (1975, 1978, hereafter referred to as SK) have proposed a model of a spin glass which apparently allows an exact solution. Interest in this model started with the physical idea that its solution is of a mean-field type. The Hamiltonian of the SK model of Ising spin glass is

$$H = -\sum_{ij} J_{ij} S_i S_j \tag{1.1}$$

for $\mathcal{N}$ Ising spin S_i. The bond interactions J_{ij} are taken as independent random variables with a mean value $J_0 = 0$ and of variance $J/\sqrt{\mathcal{N}}$. In the following, the energy unit is fixed by the choice $J = 1$. By using the replica trick, Sherrington and Kirkpatrick (1975, 1978) expressed the free energy per spin in the form

$$\beta F = -\tfrac{1}{4}\beta^2 - \lim_{n\to 0}\frac{1}{n}\max\left[-\tfrac{1}{2}\beta^2\sum_{\alpha<\beta} Q^2 + \ln \operatorname{Tr}\exp\left(\beta^2\sum_{\alpha<\beta} Q_{\alpha\beta}S_\alpha S_\beta + \beta H\sum_\alpha S_\alpha\right)\right] \tag{1.2}$$

where the indices α, β run from 1 to n and the trace is over the 2^n values of the $s_\alpha = \pm 1$. The maximum is taken over all possible matrices $Q_{\alpha\beta}$. SK presented the solution which preserves replica symmetry: $Q_{\alpha\beta} = q$. Unfortunately the SK solution suffers from a number of defects. First, the entropy becomes negative at sufficiently low temperatures. Second, de Almeida and Thouless (1978, referred to as AT) found that the matrix $\partial^2 F/\partial Q_{\alpha\beta}\,\partial Q_{\gamma\nu}$ has a negative eigenvalue. This is interpreted by AT as an instability, which may break the replica symmetry for $T < T_c$ ($T_c = 1$). To remove these defects Blandin (1978) and Bray and Moore (1978, 1979) have presented several schemes for breaking the the replica symmetry. Another replica-breaking scheme has been proposed by Parisi (1979a, b, 1980a, b, c). In Parisi's scheme, the order parameter at a given temperature is a function $q(x)$, $0 < x < 1$, which is expected to be continuous and monotonically non-decreasing. The function $q(x)$ can be found by maximising an effective free energy F which functionally depends on $q(x)$:

$$F = \max_{q(x)} F[q]. \qquad (1.3)$$

Study of the stability of Parisi's solution was begun by Thouless *et al* (1980). They considered the quadratic terms in the variation of the free energy and obtained that the operator K defined on $L^2(0, 1)$ by

$$F[q_0+\delta q]=F[q_0]+\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}x' K(x, x')\, \delta q(x)\, \delta(x')+\mathrm{O}[(q)^3] \qquad (1.4)$$

where $q_0(x)$ maximises equation (1.3), has zero as the accumulation point for its negative eigenvalues.

To examine the question of the stability of Parisi's solution we shall consider the matrix $M_{\alpha\beta,\gamma\nu}$ defined by

$$F[Q^{(0)}_{\alpha\beta}+R_{\alpha\beta}]=F[Q^{(0)}_{\alpha\beta}]+\sum_{\alpha<\beta}\sum_{\gamma<\nu} R_{\alpha\beta}M_{\alpha\beta,\gamma\nu}R_{\gamma\nu}+\mathrm{O}(R^3) \qquad (1.5)$$

where $Q^{(0)}_{\alpha\beta}$ is the stationary point used by Parisi (1979b, 1980a, b) and the functional $F[Q_{\alpha\beta}]$ is equal to the expression in square brackets in equation (1.2). This quadratic form should be positive definite for a stable solution of the problem. It may be interesting to note that the quadratic form defined by equation (1.4) should be negative definite at the same time. In § 2 we calculate some of the eigenvalues of the matrix $M_{\alpha\beta,\gamma\nu}$ and obtain the condition for stability of Parisi's solution. This condition is analysed for T close to T_c in the condensed phase (§ 3).

2. The condition for stability of Parisi's solution

Parisi (1979b) suggested the following parametrisation of the matrix $Q_{\alpha\beta}$

$$\begin{aligned} &Q_{\alpha\alpha}=0 \\ &Q_{\alpha\beta}=q_i \qquad \text{if} \quad I(\alpha/m_i)\neq I(\beta/m_i) \\ &\qquad\qquad\qquad\quad I(\alpha/m_{i+1})=I(\beta/m_{i+1}) \end{aligned} \qquad (2.1)$$

where the m_i are integers such that m_{i+1}/m_i is an integer ($i=0, k$) with $m_0=1$ and $m_{k+1}=n$; $I(x)$ is an integer-valued function: its value is the smallest integer greater than or equal to x. The matrix $Q_{\alpha\beta}$ depends on $k+1$ real parameters (m_i). If n is not a positive integer, there is no reason to have integer m_i; in the most interesting case, they satisfy (for $n=0$) the inequalities

$$1 \geqslant m_1 \geqslant m_2 \geqslant \ldots m_k \geqslant m_{k+1}=0.$$

The function $q(x)$ is defined by

$$q(x)=q_i \qquad \text{for} \quad m_i>x>m_{i+1} \qquad (i=0, k). \qquad (2.2)$$

For Parisi's parametrisation of the matrix $Q_{\alpha\beta}$, after some algebra we can write the quadratic form defined by equation (1.5) in the form

$$\sum_{\alpha<\beta}\sum_{\gamma<\nu} R_{\alpha\beta}M_{\alpha\beta,\gamma\nu}R_{\gamma\nu}$$

$$=\beta^4\left(q_0-\frac{1-T^2}{2}\right)\sum_{\alpha<\beta}R^2_{\alpha\beta}-\tfrac{1}{2}\beta^4\sum_{j=0}^{k}\tilde{q}_j\sum_{i=1}^{n/m_j+1}\sum_{\beta}\left(\sum_{\substack{\alpha(<\beta)\\ \alpha\in I_{j+1}^{(i)}}}R_{\alpha\beta}+\sum_{\substack{\alpha(>\beta)\\ \alpha\in I_{j+1}^{(i)}}}R_{\beta\alpha}\right)^2$$

$$+\tfrac{1}{2}\beta^4\left(\sum_{j=0}^{k}q_j\sum_{(\alpha\beta)\in L_j}R_{\alpha\beta}\right)^2-\tfrac{1}{2}\beta^4\sum_{\substack{\alpha<\beta\\ \alpha\neq\beta\neq\gamma\neq\nu}}\sum_{\gamma<\nu}R_{\alpha\beta}R_{\gamma\nu}\langle S_\alpha S_\beta S_\gamma S_\nu\rangle$$

where $\tilde{q}_j=q_j-q_{j+1}$, $q_{k+1}=0$; $I_j^{(i)}$ is a set of integers α which satisfy the inequality $(i-1)m_j<\alpha\leqslant im_j$ and L_j is a set of a pair $(\alpha\beta)$ for which $Q_{\alpha\beta}=q_j$. The correlation function

$$K_{\alpha\beta\gamma\nu}\equiv\langle S_\alpha S_\beta S_\gamma S_\nu\rangle \qquad (\alpha\neq\beta\neq\gamma\neq\nu) \tag{2.4}$$

is symmetric under permutation of the indices α, β, γ, ν. The parametrisation of the matrix $Q_{\alpha\beta}$ given by equation (2.1) yields the following useful property of the $K_{\alpha\beta\gamma\nu}$

$$K_{\alpha\beta\gamma\nu}=K_{\alpha'\beta'\gamma'\nu'} \qquad \text{if} \quad \alpha,\alpha'\in I_1^{(a)} \quad \beta,\beta'\in I_1^{(b)}$$
$$\gamma,\gamma'\in I_1^{(c)} \quad \nu,\nu'\in I_1^{(d)} \tag{2.5}$$
$$(\alpha\neq\beta\neq\gamma\neq\nu,\ \alpha'\neq\beta'\neq\gamma'\neq\nu')$$

where the integer numbers a, b, c and d are arbitrary. Next we consider the vectors $R^{(j)}_{\alpha\beta}$ $(j=0,k)$ which satisfy the conditions

$$R^{(j)}_{\alpha\beta}=R^{(j)}_{\beta\alpha}\begin{cases}\neq 0 & \text{for } (\alpha\beta)\in L_j\\ =0 & \text{for } (\alpha\beta)\notin L_j\end{cases} \tag{2.6a}$$

$$\sum_{\alpha(\neq\beta)\in I_1^{(a)}}R^{(j)}_{\alpha\beta}=0 \qquad \text{for all } \beta=1,n \text{ and } a=1,n/m_1. \tag{2.6b}$$

By using the property of equation (2.5) it may be proved exactly that the vectors $R^{(j)}_{\alpha\beta}$ are the eigenvectors of the matrix $M_{\alpha\beta,\gamma\nu}$ associated with the quadratic form (equation (2.3))

$$\sum_{\gamma<\nu}M_{\alpha\beta,\gamma\nu}R^{(j)}_{\gamma\nu}=\lambda_{rj}R^{(j)}_{\alpha\beta}$$
$$\lambda_{rj}=\beta^4[q_0-\tfrac{1}{2}(1-T^2)-\tfrac{1}{2}K_j] \qquad (j=0,k) \tag{2.7}$$

where

$$K_j\equiv K_{\alpha\beta\gamma\nu} \qquad \text{for } \alpha,\gamma\in I_1^{(a)} \qquad \beta,\nu\in I_1^{(b)} \tag{2.8}$$

and the integer numbers a, $b=1,n/m_1$ satisfy the condition $(\alpha\beta),(\gamma\nu)\in L_j$.

For the SK solution which preserves replica symmetry, eigenvalues λ_{rj} are equal to the eigenvalue λ_r of the replicon eigenvectors. The $n(n-3)/2$-dimensional subspace of the entire $n(n-1)/2$-dimensional space which is spanned by the replicon eigenvectors is the so-called 'replicon subspace' (Bray and Moore 1979).

Let us return to Parisi's solution. We also term the eigenvector which gives rise to the eigenvalue λ_{rj} the 'replicon eigenvector' (see equation (2.7)). There are n other eigenvectors ($n-1$ 'anomalous' eigenvectors and one 'breathing' eigenvector). Unfortunately, we can only calculate these eigenvalues for T close to T_c. For $n=0$

and $k \to \infty$ the eigenvalues λ_{rj} form the 'replicon' branch

$$\lambda_r(x) = \beta^4[q(1) - \tfrac{1}{2}(1 - T^2) - \tfrac{1}{2}K(x)] \tag{2.9}$$

where the function $K(x)$ is defined by

$$K(x) = K_j \qquad \text{if} \quad m_j > x > m_{j+1}. \tag{2.10}$$

The function $K(x)$ is expected to be continuous and monotonically non-decreasing:

$$\max K(x) = K(1) \qquad \min K(x) = K(0). \tag{2.11}$$

The condition that the eigenvalues given by equation (2.9) are positive can be written in the form

$$T^2 \geqslant 1 - 2q(1) + K(1). \tag{2.12}$$

For the SK solution the stability condition (2.12) coincides with the one obtained by AT.

Our aim is to compute the right-hand side of the inequality (2.12). Introducing a quantity

$$G = \mathrm{Tr}\exp\left(\tfrac{1}{2}\beta^2 \sum_{\alpha,\beta} Q_{\alpha\beta}S_\alpha S_\beta + \sum_\alpha h_\alpha S_\alpha\right) \tag{2.13}$$

where the indices α, β run from 1 to n we write for zero magnetic field

$$Q_{\alpha\beta} = G^{-1}\frac{\partial^2}{\partial h_\alpha\, \partial h_\beta}G\bigg|_{h_\alpha = h = 0} \tag{2.14a}$$

$$K_{\alpha\beta\gamma\nu} = G^{-1}\frac{\partial^4}{\partial h_\alpha\, \partial h_\beta\, \partial h_\gamma\, \partial h_\nu}G\bigg|_{h_\alpha = h = 0} \tag{2.14b}$$

By using the method proposed by Duplantier (1981) we obtain

$$1 - 2q(1) + K(1) = \Delta(0, 0) \tag{2.15}$$

where the function $\Delta(x, h)$ satisfies the following differential equation

$$\frac{\partial}{\partial x}\Delta(x, h) = -\tfrac{1}{2}\beta^2\frac{\mathrm{d}q(x)}{\mathrm{d}x}\frac{\partial^2}{\partial h^2}\Delta(x, h) + \Delta(x, h)f(x, h) \tag{2.16}$$

with the boundary condition

$$\Delta(1, h) = 2\,\mathrm{sech}^3\, h.$$

The function $f(x, h)$ satisfies the equation obtained by Parisi (1980b):

$$\frac{\partial f}{\partial x} = -\tfrac{1}{2}\beta^2\frac{\mathrm{d}q(x)}{\mathrm{d}x}\left[\frac{\partial^2 f}{\partial h^2} + x\left(\frac{\partial f}{\partial h}\right)^2\right] \tag{2.17}$$

$f(1, h) = \ln(2\cosh h)$.

Equation (2.15) is correct as it stands only if $q(0) = 0$, otherwise we would have

$$1 - 2q(1) + K(1) = \int_{-\infty}^{\infty}\frac{\mathrm{d}z}{\sqrt{2\pi}}\mathrm{e}^{-z^2/2}\Delta(0, \beta H + \beta z\sqrt{q(0)}) \tag{2.18}$$

where H is a magnetic field. Use of equations (2.16)–(2.18) for the SK solution yields the AT result.

3. Stability of Parisi's solution near T_c

Let us consider the stability of Parisi's solution for T close to T_c in zero magnetic field. The free energy (equation (1.2)) takes the form (Pytte and Rudnick 1979)

$$\beta F = -\ln 2 - \tfrac{1}{4}\beta^2 - \lim_{n\to 0}\frac{1}{n}\Big(\tfrac{1}{4}\beta^4(1-T^2)\sum_{\alpha,\beta} Q_{\alpha\beta}^2 + \tfrac{1}{6}\beta^6 \sum_{\alpha,\beta,\gamma} Q_{\alpha\beta}Q_{\beta\gamma}Q_{\alpha\gamma}$$
$$+\frac{\beta^8}{8}\sum_{\alpha,\beta,\gamma,\nu} Q_{\alpha\beta}Q_{\beta\gamma}Q_{\gamma\nu}Q_{\nu\alpha} - \tfrac{1}{4}\beta^8\sum_{\alpha,\beta,\gamma} Q_{\alpha\beta}^2Q_{\alpha\gamma}^2 + \tfrac{1}{12}\sum_{\alpha,\beta} Q_{\alpha\beta}^4\Big) + \mathrm{O}(Q^5) \tag{3.1}$$

where sums over the replica indices are unrestricted except that $Q_{\alpha\beta}=0$ for $\alpha=\beta$. The free energy (equation (3.1)) may be written as a functional of $q(x)$ (Thouless *et al* 1980). The function $q(x)$ which maximises equation (3.1) has the form

$$q(x) = \begin{cases} q(1) & x_1 \leqslant x \leqslant 1 \\ \tfrac{1}{2}(1+3\tau)x + \mathrm{O}(\tau^3) & 0 \leqslant x < x_1 \end{cases} \tag{3.2a}$$

$$q(1) = \tau + \tau^2 + \mathrm{O}(\tau^3) \tag{3.2b}$$

where $\tau = 1 - T/T_c$. Use of equations (2.4) and (2.8) gives

$$K(x) = q^2(1) + 2q^2(x) + \mathrm{O}(q^3). \tag{3.3}$$

Hence

$$\lambda_r(x) = \beta^4[q(1) - \tfrac{1}{2}(1-T^2) - \tfrac{1}{2}q^2(1) - q^2(x) + \mathrm{O}(q^3)]. \tag{3.4}$$

Substituting equations (3.2) and (3.3) in equation (3.4) we obtain at second order in τ

$$\lambda_r(1) = 0. \tag{3.5}$$

The eigenvalues $\lambda_r(x)$ are non-negative and the stability condition (2.12) is satisfied to this order. It may be interesting to note that for the SK solution the inequality (2.12) is violated by the terms of order τ^2. Unfortunately we do not know whether the inequality (2.12) is satisfied at the third and all higher orders in τ.

Parisi (1980a, c) has taken a simpler form for the quartic terms in equation (3.1), so that he has

$$\beta F = -\ln 2 - \tfrac{1}{4}\beta^2 - \lim_{n\to 0}\frac{1}{n}\Big(\tau\sum_{\alpha<\beta} Q_{\alpha\beta}^2 + \sum_{\alpha<\beta<\gamma} Q_{\alpha\beta}Q_{\beta\gamma}Q_{\alpha\gamma} + \tfrac{1}{6}\sum_{\alpha<\beta} Q_{\alpha\beta}^4\Big). \tag{3.6}$$

Substituting equation (3.6) in equation (1.5), for Parisi's parametrisation of the matrix $Q_{\alpha\beta}$ one finds that the quadratic form defined by equation (1.5) has the form

$$\sum_{\alpha<\beta}\sum_{\gamma<\nu} R_{\alpha\beta}M_{\alpha\beta,\gamma\nu}R_{\gamma\nu}$$
$$= \sum_{\alpha<\beta}(q_0 - \tau - Q_{\alpha\beta}^2)R_{\alpha\beta}^2 - \tfrac{1}{2}\sum_{j=0}^{k} q_j \sum_{i=1}^{n/m_{j+1}}\sum_{\beta}\Big(\sum_{\alpha(\neq\beta)\in I_{j+1}^{(i)}} R_{\alpha\beta}\Big)^2. \tag{3.7}$$

The eigenvalues of the matrix $M_{\alpha\beta,\gamma\nu}$ may be found from the eigenvalue equation

$$\det[\lambda\delta_{\alpha\gamma}\delta_{\beta\nu} - M_{\alpha\beta,\gamma\nu}] = 0. \tag{3.8}$$

Using the identity

$$\det{}^{-1/2}[\lambda\delta_{\alpha\gamma}\delta_{\beta\nu} - M_{\alpha\beta,\gamma\nu}] = \int_{-\infty}^{\infty} \prod_{\alpha<\beta} \frac{\mathrm{d}R_{\alpha\beta}}{\sqrt{2\pi}} \exp\Big(-\tfrac{1}{2} \sum_{\alpha<\beta} \sum_{\gamma<\nu} R_{\alpha\beta}(\lambda\delta_{\alpha\gamma}\delta_{\beta\nu} - M_{\alpha\beta,\gamma\nu})R_{\gamma\nu}\Big) \tag{3.9}$$

and calculating the integral in the right-hand side of this identity by means of the Hubbard–Stratonovich transformation, one obtains

$$\det[\lambda\delta_{\alpha\gamma}\delta_{\beta\nu} - M_{\alpha\beta,\gamma\nu}] = \Big(\prod_{j=0}^{k} (\lambda - \lambda_{rj})^{n(m_{j+1}-m_j)/2}(1 - D_j(\lambda))^{n(m_j^{-1}-m_{j+1}^{-1})}\Big)(1 - D_{k+1}(\lambda)) \tag{3.10}$$

where

$$\lambda_{rj} = q_0 - \tau - q_j^2 \qquad (j = 0, k) \tag{3.11}$$

$$D_j(\lambda) = 1 - \tfrac{1}{2}q_0 \sum_{i=0}^{k} \frac{m_i - m_{i+1}}{\lambda - \lambda_{ri}} - \tfrac{1}{2}q_0 \sum_{i=0}^{j-1} \frac{m_i - m_{i+1}}{\lambda - \lambda_{ri}} - \frac{q_0 m_j}{2(\lambda - \lambda_{rj})} \qquad (j = 0, k) \tag{3.12a}$$

$$D_{k+1}(\lambda) = 1 - q_0 \sum_{i=0} \frac{m_i - m_{i+1}}{\lambda - \lambda_{ri}}. \tag{3.12b}$$

From equations (3.8) and (3.10) we have that there are $k+1$ eigenvalues λ_{rj}, $(j = 0, k)$. Using equation (3.7) one can prove that the eigenvector which satisfies equation (2.6) gives rise to the eigenvalue λ_{rj}. There are $k+2$ eigenvalues $(j = 0, k+1)$ given by the equations

$$D_j(\lambda_{aj}) = 0 \qquad (j = 0, k+1) \tag{3.13}$$

where $D_j(\lambda)$ is determined by equations (3.12).

If the replica symmetry is restored, the eigenvalues λ_{rj} and λ_{aj} are given by

$$\lambda_{rj} = \lambda_r = q - \tau - q^2 = -\tfrac{2}{3}\tau^2 + O(\tau^3) \tag{3.14a}$$

$$\lambda_{aj} = \lambda_a = \lambda_b = 2q - \tau - q^2 = \tau - \tfrac{1}{3}\tau^2 + O(\tau^3) \tag{3.14b}$$

where $q = \tau + \frac{1}{3}\tau^2 + O(\tau^3)$. λ_r is the eigenvalue of the replicon eigenvector and λ_a and λ_b are the eigenvalues of the 'anomalous' eigenvectors and 'breathing' eigenvectors accordingly.

For $n = 0$ and $k \to \infty$ equations (3.11) and (3.13) have the form

$$\lambda_r(x) = q(1) - \tau - q^2(x) \tag{3.15}$$

$$1 - \tfrac{1}{2}q(1)\int_0^1 \frac{\mathrm{d}y}{\lambda_a(x) - \lambda_r(y)} - \tfrac{1}{2}q(1)\int_x^1 \frac{\mathrm{d}y}{\lambda_a(x) - \lambda_r(y)} - \frac{xq(1)}{2(\lambda_a(x) - \lambda_r(x))} = 0. \tag{3.16}$$

For the free energy (equation (3.6)) the function $q(x)$ is

$$q(x) = \begin{cases} q(1) & x_1 \leqslant x \leqslant 1 \\ \tfrac{1}{2}x & 0 \leqslant x < x_1 \end{cases} \tag{3.17a}$$

$$q(1) = \tau + q^2(1) \qquad x_1 = 2q(1). \tag{3.17b}$$

(Parisi 1980c). Substituting equations (3.17) into equation (3.15) one finds

$$\lambda_r(x)=\begin{cases}\lambda_r(1)=0 & x_1\leqslant x\leqslant 1\\ \frac{1}{2}(x_1^2-x^2) & 0\leqslant x<x_1.\end{cases}\qquad(3.18)$$

It may be interesting to note that for the approximate free energy (equation (3.6)) $\lambda_r(1)$ is equal to zero, exactly. Solving equation (3.16), we obtain

$$\lambda_a(x)=\begin{cases}\tau+\tau^2+\frac{8}{3}\tau^3+O(\tau^4) & x_1\leqslant x\leqslant 1\\ \tau+\tau^2+\frac{10}{3}\tau^3-\frac{1}{12}x^3+O(\tau^4) & 0\leqslant x<x_1\end{cases}\qquad(3.19)$$

So we obtain that for Parisi's solution all eigenvalues ($\lambda_r(x)$ and $\lambda_a(x)$) of the matrix $M_{\alpha\beta,\gamma\nu}$ are non-negative and the stability condition (equation (2.12)) is satisfied up to the second order in τ for T close to T_c.

4. Conclusions

In the present paper we proved that the condition for stability of the replica-symmetry-breaking solution proposed by Parisi is determined by the inequality (2.12). This inequality is satisfied for T close to T_c up to the second order in $\tau=1-T/T_c$. For the approximate free energy this inequality is satisfied at all orders in τ. To examine the stability of Parisi's solution at low temperatures one has to solve the differential equations (2.16) and (2.17).

We have shown that there are zero eigenvalues of the matrix $M_{\alpha\beta,\gamma\nu}$ of second derivatives. Other eigenvalues are positive. These results are correct to second order in τ. The breaking of the replica symmetry partially removes a degeneracy of the replicon and anomalous eigenvalues. There are two continuous branches of the eigenvalues of the matrix $M_{\alpha\beta,\gamma\nu}$. One of them is formed by the replicon eigenvalues and the other one is formed by the anomalous and breather eigenvalues.

This paper presents further evidence that Parisi's solution may be the correct one. However, there are important questions unsolved. For example, a question of great interest is whether the model contains massless modes in all orders in perturbation theory.

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